On the Extension of the TurboMOR-RC Reduction Method to RLC Circuits

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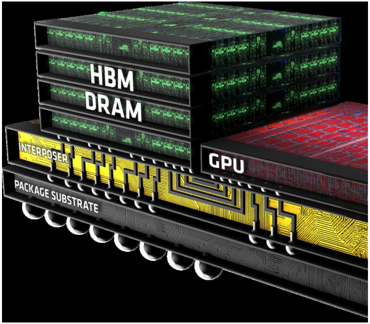
Motivation: Simulation of modern VLSI systems

- Higher data rates
- Lower voltages
- Higher level of integration

- Parasitic electromagnetic effects become more and more important
- Need to be included in the simulation

short circuit $\rightarrow$ RLC network



Source: amd.com

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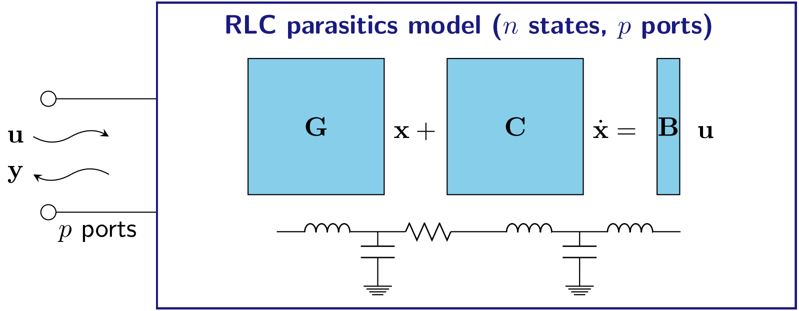
Models of interconnect parasitics

Netlist	Number of nodes	Number of ports
PLL RC parasitics[1]	381k	4k
Receiver RC parasitics[1]	803k	15k
3D-IC power grid[2]	9M	3.3M

[1] Ionuţiu, Rommes, & Schilders (2011)

[2] P.-W. Luo *et al.* (2013)

Model order reduction (MOR) via moment matching



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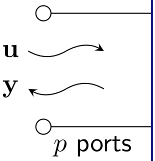

Reduced model ($\tilde{n}$ states $\ll n$, p ports)

$$\tilde{\mathbf{G}}\,\tilde{\mathbf{x}} + \tilde{\mathbf{C}}\,\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{B}}\,\mathbf{u}$$

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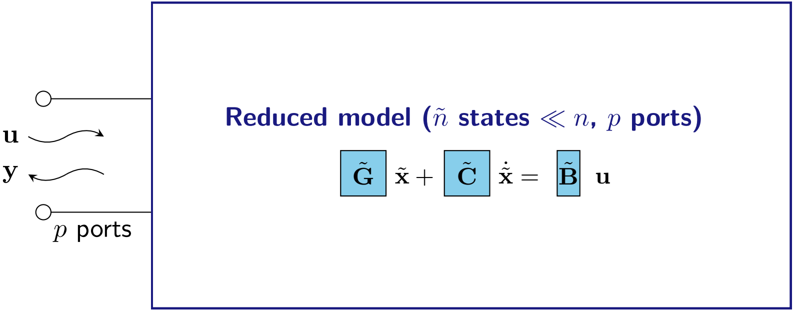

Reduced model approximates the original by matching the first q moments around an expansion point (e.g. DC: $s_0 = 0$)

$$\mathbf{H}(s) = \mathbf{M}_0 + \mathbf{M}_1 s + \mathbf{M}_2 s^2 + \cdots + \mathbf{M}_{q-1} s^{q-1} + \mathbf{M}_q s^q + \ldots$$

$$\tilde{\mathbf{H}}(s) = \underbrace{\mathbf{M}_0 + \mathbf{M}_1 s + \mathbf{M}_2 s^2 + \cdots + \mathbf{M}_{q-1} s^{q-1}}_{\text{matched}} + \underbrace{\hat{\mathbf{M}}_q s^q + \ldots}_{\text{not matched}}$$

State of the art: PRIMA

- System with n states and p ports ($p \ll n$)

Odabasioglu, Celik, & Pileggi (1998)

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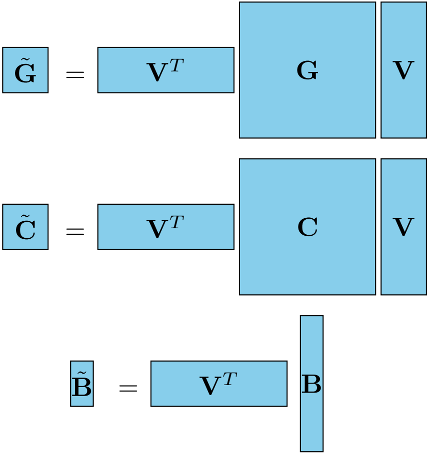

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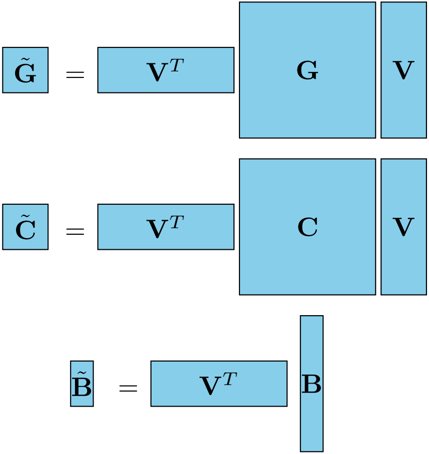

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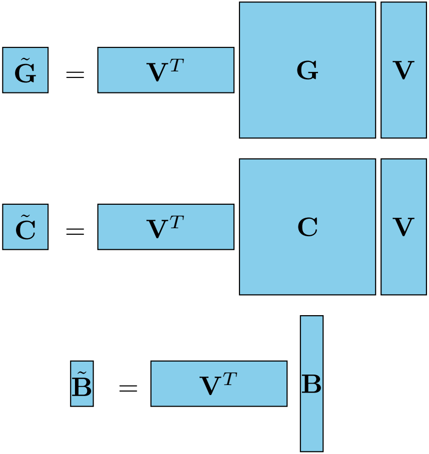

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- Time-consuming to carry out the projection

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Some acceleration approaches and challenges

Approach	Works	Challenges
Node elimination based on time constants	TICER (RC) [Sheehan; 1999] RLC technique [Amin *et al.*; 2005]	• Effectiveness is case-specific
Partitioning	BVOR [Yu, *et al.*; 2006] SparseRC [Ionuțiu; 2011] PartMOR [Miettinen *et al.*; 2011]	• RLC(K) equations are harder to partition than RC
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Efficient reduction becomes much more difficult once you introduce inductors into the model.

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Goal: Extend TurboMOR-RC to RLC case where this assumption is violated

Original system

MNA equations:

$\mathbf{x}_1$: port-related unknowns (p)

$\mathbf{x}_2$: all other unknowns $(n-p)$

$$\begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{21}^T \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

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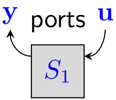

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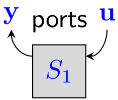

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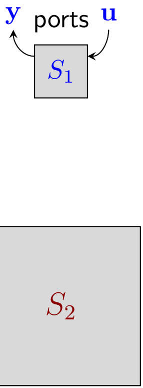

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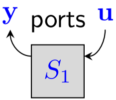

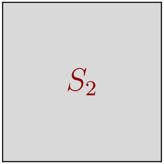

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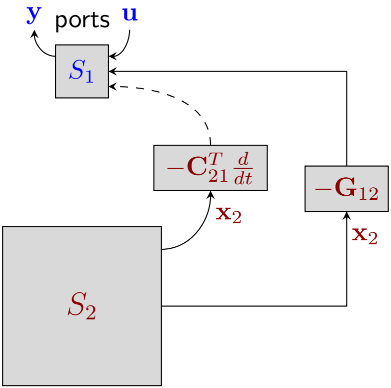

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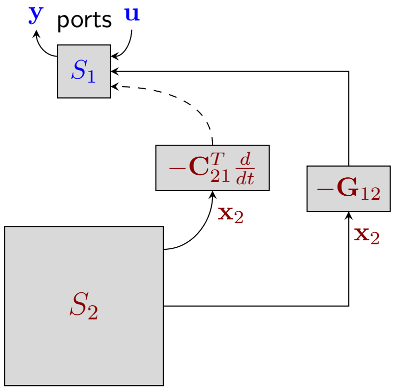

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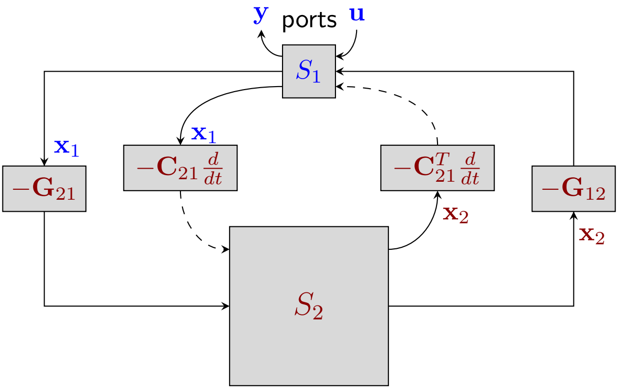

Iteration 1: matching one moment at DC

Congruence with $\mathbf{M}^{(1)} = \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \begin{array}{c}\mathbf{0} \\ \mathbf{M}_2^{(1)}\end{array} \end{array}\right] = \left[\begin{array}{c|c} \begin{array}{c}\mathbf{I} \\ -\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\end{array} & \begin{array}{c}\mathbf{0} \\ \mathbf{G}_{22}^{-T}\end{array} \end{array}\right]$

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$-\mathbf{G}_{22}^{-1}\mathbf{G}_{21}$ eliminates $\mathbf{G}_{21}$

Iteration 1: matching one moment at DC

Congruence with $\mathbf{M}^{(1)} = \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \begin{array}{c}\mathbf{0}\\ \mathbf{M}_2^{(1)}\end{array} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{22}^{-1}\mathbf{G}_{21} & \mathbf{G}_{22}^{-T} \end{array}\right]$

$$\mathbf{G}^{(1)} = \left[\begin{array}{cc} \multicolumn{2}{c}{\mathbf{M}_1^{(1)T}} \\ \hline \mathbf{0} & \mathbf{M}_2^{(2)T} \end{array}\right] \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \begin{array}{c}\mathbf{0}\\ \mathbf{M}_2^{(1)}\end{array} \end{array}\right] = \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix}$$

$-\mathbf{G}_{22}^{-1}\mathbf{G}_{21}$ eliminates $\mathbf{G}_{21}$

$$\mathbf{C}^{(1)} = \left[\begin{array}{cc} \multicolumn{2}{c}{\mathbf{M}_1^{(1)T}} \\ \hline \mathbf{0} & \mathbf{M}_2^{(2)T} \end{array}\right] \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{21}^{T} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \begin{array}{c}\mathbf{0}\\ \mathbf{M}_2^{(1)}\end{array} \end{array}\right] = \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(1)T} \\ \mathbf{C}_{21}^{(1)} & \mathbf{C}_{22}^{(1)} \end{bmatrix}$$

Iteration 1: matching one moment at DC

Congruence with $\mathbf{M}^{(1)} = \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \begin{matrix}\mathbf{0} \\ \mathbf{M}_2^{(1)}\end{matrix} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{22}^{-1}\mathbf{G}_{21} & \mathbf{G}_{22}^{-T} \end{array}\right]$

$$\mathbf{G}^{(1)} = \left[\begin{array}{cc} \multicolumn{2}{c}{\mathbf{M}_1^{(1)T}} \\ \hline \mathbf{0} & \mathbf{M}_2^{(2)T} \end{array}\right] \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \begin{matrix}\mathbf{0} \\ \mathbf{M}_2^{(1)}\end{matrix} \end{array}\right] = \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix}$$

$-\mathbf{G}_{22}^{-1}\mathbf{G}_{21}$ eliminates $\mathbf{G}_{21}$

$$\mathbf{C}^{(1)} = \left[\begin{array}{cc} \multicolumn{2}{c}{\mathbf{M}_1^{(1)T}} \\ \hline \mathbf{0} & \mathbf{M}_2^{(2)T} \end{array}\right] \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{21}^{T} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \begin{matrix}\mathbf{0} \\ \mathbf{M}_2^{(1)}\end{matrix} \end{array}\right] = \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(1)T} \\ \mathbf{C}_{21}^{(1)} & \mathbf{C}_{22}^{(1)} \end{bmatrix}$$

$$\mathbf{B}^{(1)} = \left[\begin{array}{cc} \multicolumn{2}{c}{\mathbf{M}_1^{(1)T}} \\ \hline \mathbf{0} & \mathbf{M}_2^{(2)T} \end{array}\right] \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}$$

Iteration 1: matching one moment at DC

Congruence with $\mathbf{M}^{(1)} = \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \mathbf{0} \\ & \mathbf{M}_2^{(1)} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ -\mathbf{G}_{22}^{-1}\mathbf{G}_{21} & \mathbf{G}_{22}^{-T} \end{array}\right]$

$$\mathbf{G}^{(1)} = \left[\begin{array}{c} \mathbf{M}_1^{(1)T} \\ \hline \mathbf{0} \quad \mathbf{M}_2^{(2)T} \end{array}\right] \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \mathbf{0} \\ & \mathbf{M}_2^{(1)} \end{array}\right] = \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix}$$

$-\mathbf{G}_{22}^{-1}\mathbf{G}_{21}$ eliminates $\mathbf{G}_{21}$

$$\mathbf{C}^{(1)} = \left[\begin{array}{c} \mathbf{M}_1^{(1)T} \\ \hline \mathbf{0} \quad \mathbf{M}_2^{(2)T} \end{array}\right] \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{21}^T \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \mathbf{0} \\ & \mathbf{M}_2^{(1)} \end{array}\right] = \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(1)T} \\ \mathbf{C}_{21}^{(1)} & \mathbf{C}_{22}^{(1)} \end{bmatrix}$$

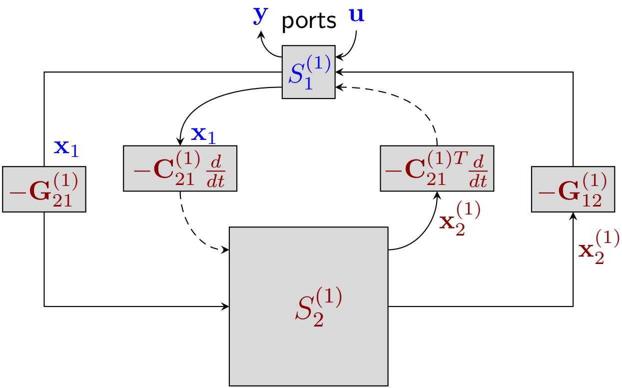
$$\mathbf{B}^{(1)} = \left[\begin{array}{c} \mathbf{M}_1^{(1)T} \\ \hline \mathbf{0} \quad \mathbf{M}_2^{(2)T} \end{array}\right] \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}$$

The zeros in $\mathbf{B}$ are preserved after $\mathbf{M}^{(1)T}\mathbf{B}$

Iteration 1: matching one moment at DC

$$\begin{matrix} S_1^{(1)}: \\ S_2^{(2)}: \end{matrix} \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(1)T} \\ \mathbf{C}_{21}^{(1)} & \mathbf{C}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

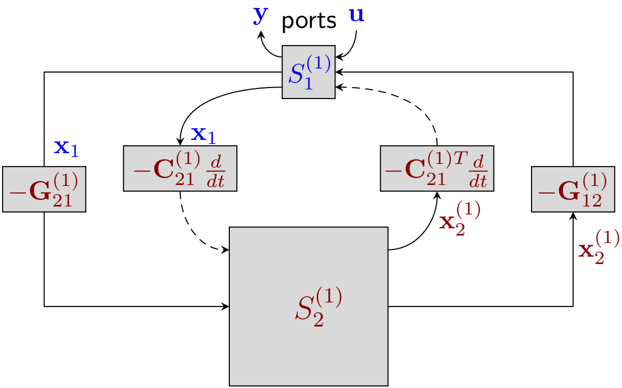
$$\mathbf{y} = \frac{1}{2} \begin{bmatrix} \mathbf{B}_1^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} - \mathbf{u}$$



Iteration 1: matching one moment at DC

$$\begin{matrix} S_1^{(1)}: \\ S_2^{(2)}: \end{matrix} \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(1)T} \\ \mathbf{C}_{21}^{(1)} & \mathbf{C}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \frac{1}{2} \begin{bmatrix} \mathbf{B}_1^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} - \mathbf{u}$$



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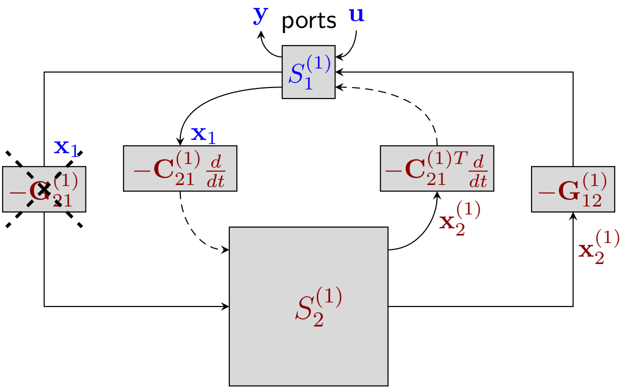
Iteration 1: matching one moment at DC

$$\begin{matrix} S_1^{(1)}: \\ S_2^{(2)}: \end{matrix} \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(1)T} \\ \mathbf{C}_{21}^{(1)} & \mathbf{C}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

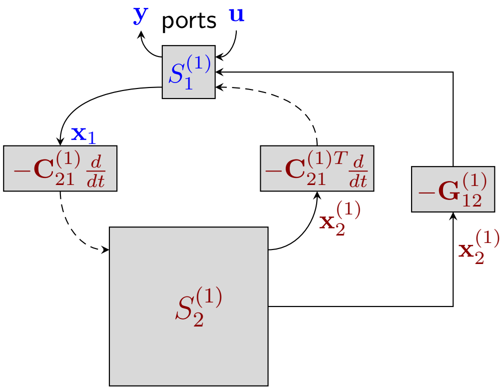
$$\mathbf{y} = \frac{1}{2} \begin{bmatrix} \mathbf{B}_1^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} - \mathbf{u}$$



Iteration 1: matching one moment at DC

$$\begin{matrix} S_1^{(1)}: \\ S_2^{(2)}: \end{matrix} \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(1)T} \\ \mathbf{C}_{21}^{(1)} & \mathbf{C}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

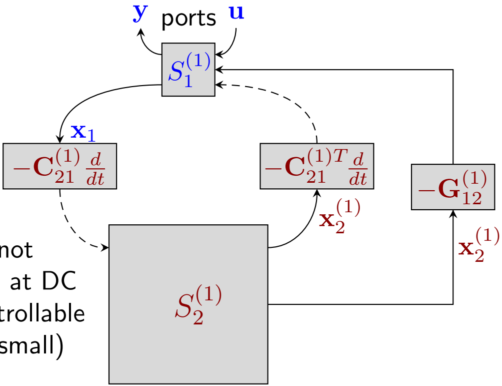
$$\mathbf{y} = \frac{1}{2} \begin{bmatrix} \mathbf{B}_1^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} - \mathbf{u}$$



Iteration 1: matching one moment at DC

$$\begin{matrix} S_1^{(1)}: \\ S_2^{(2)}: \end{matrix} \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(1)T} \\ \mathbf{C}_{21}^{(1)} & \mathbf{C}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \frac{1}{2} \begin{bmatrix} \mathbf{B}_1^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} - \mathbf{u}$$

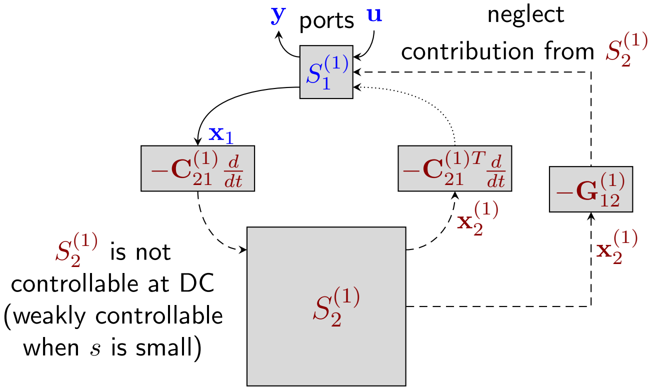


$S_2^{(1)}$ is not controllable at DC (weakly controllable when s is small)

Iteration 1: matching one moment at DC

$$\begin{matrix} S_1^{(1)}: \\ S_2^{(2)}: \end{matrix} \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(1)T} \\ \mathbf{C}_{21}^{(1)} & \mathbf{C}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

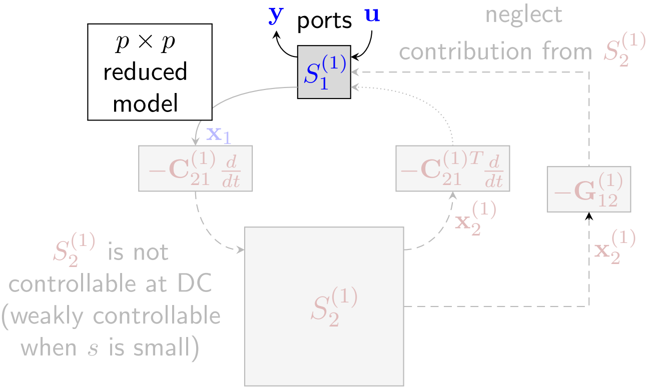
$$\mathbf{y} = \frac{1}{2} \begin{bmatrix} \mathbf{B}_1^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} - \mathbf{u}$$



Iteration 1: matching one moment at DC

$$\begin{matrix} S_1^{(1)}: \\ S_2^{(2)}: \end{matrix} \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(1)T} \\ \mathbf{C}_{21}^{(1)} & \mathbf{C}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \frac{1}{2} \begin{bmatrix} \mathbf{B}_1^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} - \mathbf{u}$$



Iteration 1: matching one moment at DC

Congruence transformation with $\mathbf{M}^{(1)} = \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \begin{matrix}\mathbf{0} \\ \mathbf{M}_2^{(1)}\end{matrix} \end{array}\right]$

$$\mathbf{G}^{(1)} = \left[\begin{array}{cc} \multicolumn{2}{c}{\mathbf{M}_1^{(1)T}} \\ \hline \mathbf{0} & \mathbf{M}_2^{(2)T} \end{array}\right] \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \begin{matrix}\mathbf{0} \\ \mathbf{M}_2^{(1)}\end{matrix} \end{array}\right] = \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix}$$

(Similarly for $\mathbf{C}^{(1)}$ and $\mathbf{B}^{(1)}$)

Iteration 1: matching one moment at DC

Congruence transformation with $\mathbf{M}^{(1)} = \left[\mathbf{M}_1^{(1)} \;\middle|\; \begin{matrix} \mathbf{0} \\ \mathbf{M}_2^{(1)} \end{matrix} \right]$

$$\mathbf{G}^{(1)} = \left[\begin{array}{cc} \multicolumn{2}{c}{\mathbf{M}_1^{(1)T}} \\ \hline \mathbf{0} & \mathbf{M}_2^{(2)T} \end{array} \right] \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \left[\mathbf{M}_1^{(1)} \;\middle|\; \begin{matrix} \mathbf{0} \\ \mathbf{M}_2^{(1)} \end{matrix} \right] = \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix}$$

(Similarly for $\mathbf{C}^{(1)}$ and $\mathbf{B}^{(1)}$)

- Only $\mathbf{G}_{11}^{(1)}$ is part of the reduced model

Iteration 1: matching one moment at DC

Congruence transformation with $\mathbf{M}^{(1)} = \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \begin{array}{c} \mathbf{0} \\ \mathbf{M}_2^{(1)} \end{array} \end{array}\right]$

$$\mathbf{G}^{(1)} = \left[\begin{array}{cc} \multicolumn{2}{c}{\mathbf{M}_1^{(1)T}} \\ \hline \mathbf{0} & \mathbf{M}_2^{(2)T} \end{array}\right] \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \left[\begin{array}{c|c} \mathbf{M}_1^{(1)} & \begin{array}{c} \mathbf{0} \\ \mathbf{M}_2^{(1)} \end{array} \end{array}\right] = \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix}$$

(Similarly for $\mathbf{C}^{(1)}$ and $\mathbf{B}^{(1)}$)

- Only $\mathbf{G}_{11}^{(1)}$ is part of the reduced model
- Only need the first p columns of $\mathbf{M}^{(1)}$: $\mathbf{M}_1^{(1)} = \begin{bmatrix} \mathbf{I} \\ -\mathbf{G}_{22}^{-1}\mathbf{G}_{21} \end{bmatrix}$ – *tall & thin*

Iteration 1: matching one moment at DC

Congruence transformation with $\mathbf{M}^{(1)} = \left[\mathbf{M}_1^{(1)} \;\middle|\; \begin{matrix} \mathbf{0} \\ \mathbf{M}_2^{(1)} \end{matrix} \right]$

$$\mathbf{G}^{(1)} = \begin{bmatrix} \mathbf{M}_1^{(1)T} \\ \hline \mathbf{0} \quad \mathbf{M}_2^{(2)T} \end{bmatrix} \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \left[\mathbf{M}_1^{(1)} \;\middle|\; \begin{matrix} \mathbf{0} \\ \mathbf{M}_2^{(1)} \end{matrix} \right] = \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix}$$

(Similarly for $\mathbf{C}^{(1)}$ and $\mathbf{B}^{(1)}$)

- Only $\mathbf{G}_{11}^{(1)}$ is part of the reduced model
- Only need the first p columns of $\mathbf{M}^{(1)}$: $\mathbf{M}_1^{(1)} = \begin{bmatrix} \mathbf{I} \\ -\mathbf{G}_{22}^{-1}\mathbf{G}_{21} \end{bmatrix}$ – *tall & thin*
- $\mathbf{G}_{22}^{-1}\mathbf{G}_{21}$ is computed efficiently using sparse LU factorization

Iteration 1: matching one moment at DC

Congruence transformation with $\mathbf{M}^{(1)} = \left[\mathbf{M}_1^{(1)} \,\middle|\, \begin{matrix} \mathbf{0} \\ \mathbf{M}_2^{(1)} \end{matrix} \right]$

$$\mathbf{G}^{(1)} = \left[\begin{array}{cc} \multicolumn{2}{c}{\mathbf{M}_1^{(1)T}} \\ \hline \mathbf{0} & \mathbf{M}_2^{(2)T} \end{array} \right] \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \left[\mathbf{M}_1^{(1)} \,\middle|\, \begin{matrix} \mathbf{0} \\ \mathbf{M}_2^{(1)} \end{matrix} \right] = \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix}$$

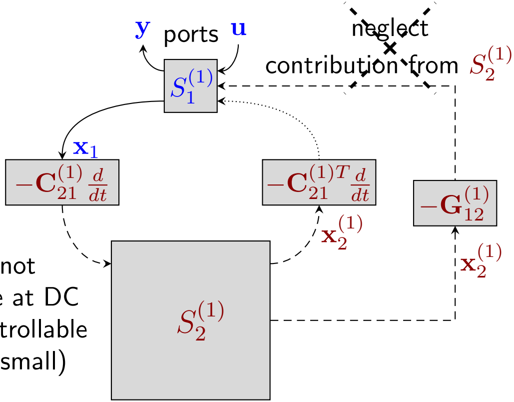
(Similarly for $\mathbf{C}^{(1)}$ and $\mathbf{B}^{(1)}$)

- Only $\mathbf{G}_{11}^{(1)}$ is part of the reduced model
- Only need the first p columns of $\mathbf{M}^{(1)}$: $\mathbf{M}_1^{(1)} = \begin{bmatrix} \mathbf{I} \\ -\mathbf{G}_{22}^{-1}\mathbf{G}_{21} \end{bmatrix}$ – *tall & thin*
- $\mathbf{G}_{22}^{-1}\mathbf{G}_{21}$ is computed efficiently using sparse LU factorization
- Note: $q = 1$ reduced model is equivalent to SIP [Ye *et al.*; 2008]

Iteration 2: matching two moments at DC

$$\begin{matrix} S_1^{(1)}: \\ S_2^{(2)}: \end{matrix} \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(1)T} \\ \mathbf{C}_{21}^{(1)} & \mathbf{C}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \frac{1}{2} \begin{bmatrix} \mathbf{B}_1^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} - \mathbf{u}$$



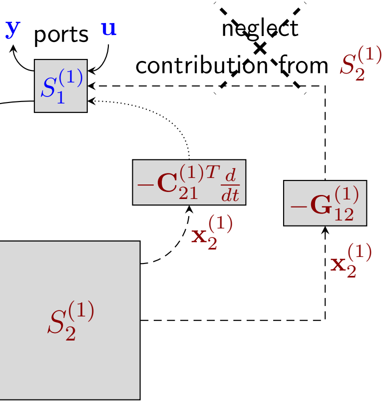
$S_2^{(1)}$ is not controllable at DC (weakly controllable when s is small)

Iteration 2: matching two moments at DC

$$\begin{matrix} S_1^{(1)}: \\ S_2^{(2)}: \end{matrix} \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(1)} \\ \mathbf{0} & \mathbf{G}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(1)T} \\ \mathbf{C}_{21}^{(1)} & \mathbf{C}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \frac{1}{2} \begin{bmatrix} \mathbf{B}_1^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(1)} \end{bmatrix} - \mathbf{u}$$

Intuition:
Retain the part of $S_2^{(1)}$ that is significant.
$\mathbf{C}_{21}^{(1)}$ plays the role of $\mathbf{B}$.



$S_2^{(1)}$ is not controllable at DC (weakly controllable when s is small)

Iteration 2: matching two moments at DC

- $\mathbf{C}_{21}^{(1)} = \underbrace{\mathbf{G}_{22}^{-1} \begin{bmatrix} \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \mathbf{M}_1^{(1)}}_{\text{from iteration 1}}$

Iteration 2: matching two moments at DC

- $\mathbf{C}_{21}^{(1)} = \underbrace{\mathbf{G}_{22}^{-1}\begin{bmatrix}\mathbf{C}_{21} & \mathbf{C}_{22}\end{bmatrix}\mathbf{M}_1^{(1)}}_{\text{from iteration 1}} \rightarrow$ QR: $\mathbf{C}_{21}^{(1)} = \begin{bmatrix}\mathbf{Q}_2^{(2)} & \mathbf{Q}_3^{(2)}\end{bmatrix}\begin{bmatrix}\mathbf{C}_{21}^{(2)} \\ \mathbf{0}\end{bmatrix} = \mathbf{Q}_2^{(2)}\mathbf{C}_{21}^{(2)}$

Iteration 2: matching two moments at DC

- $\mathbf{C}_{21}^{(1)} = \underbrace{\mathbf{G}_{22}^{-1}\begin{bmatrix}\mathbf{C}_{21} & \mathbf{C}_{22}\end{bmatrix}\mathbf{M}_1^{(1)}}_{\text{from iteration 1}} \rightarrow$ QR: $\mathbf{C}_{21}^{(1)} = \begin{bmatrix}\mathbf{Q}_2^{(2)} & \mathbf{Q}_3^{(2)}\end{bmatrix}\begin{bmatrix}\mathbf{C}_{21}^{(2)} \\ \mathbf{0}\end{bmatrix} = \mathbf{Q}_2^{(2)}\mathbf{C}_{21}^{(2)}$
- $\mathbf{M}^{(2)} = \left[\mathbf{M}_1^{(1)} \;\middle|\; \begin{matrix}\mathbf{0} \\ \mathbf{M}_2^{(2)}\end{matrix} \;\middle|\; \begin{matrix}\mathbf{0} \\ \mathbf{M}_3^{(2)}\end{matrix}\right] = \left[\mathbf{M}_1^{(1)} \;\middle|\; \begin{matrix}\mathbf{0} \\ \mathbf{Q}_2^{(2)}(\boldsymbol{\mathcal{G}}^{(2)})^{-T}\end{matrix} \;\middle|\; \begin{matrix}\mathbf{0} \\ \mathbf{G}_{22}^{-T}\mathbf{Q}_3^{(2)}\end{matrix}\right]$

Iteration 2: matching two moments at DC

- $\mathbf{C}_{21}^{(1)} = \underbrace{\mathbf{G}_{22}^{-1}\begin{bmatrix}\mathbf{C}_{21} & \mathbf{C}_{22}\end{bmatrix}\mathbf{M}_1^{(1)}}_{\text{from iteration 1}} \rightarrow$ QR: $\mathbf{C}_{21}^{(1)} = \begin{bmatrix}\mathbf{Q}_2^{(2)} & \mathbf{Q}_3^{(2)}\end{bmatrix}\begin{bmatrix}\mathbf{C}_{21}^{(2)} \\ \mathbf{0}\end{bmatrix} = \mathbf{Q}_2^{(2)}\mathbf{C}_{21}^{(2)}$
- $\mathbf{M}^{(2)} = \left[\begin{array}{c|c|c}\mathbf{M}_1^{(1)} & \begin{matrix}\mathbf{0} \\ \mathbf{M}_2^{(2)}\end{matrix} & \begin{matrix}\mathbf{0} \\ \mathbf{M}_3^{(2)}\end{matrix}\end{array}\right] = \left[\begin{array}{c|c|c}\mathbf{M}_1^{(1)} & \begin{matrix}\mathbf{0} \\ \mathbf{Q}_2^{(2)}(\boldsymbol{\mathcal{G}}^{(2)})^{-T}\end{matrix} & \begin{matrix}\mathbf{0} \\ \mathbf{G}_{22}^{-T}\mathbf{Q}_3^{(2)}\end{matrix}\end{array}\right]$

$$\mathbf{G}^{(1)} = \begin{bmatrix}\mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{33}^{(2)}\end{bmatrix} \quad \mathbf{C}^{(1)} = \begin{bmatrix}\mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} & \mathbf{0} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} & \mathbf{C}_{32}^{(2)T} \\ \mathbf{0} & \mathbf{C}_{32}^{(2)} & \mathbf{C}_{33}^{(2)}\end{bmatrix} \quad \mathbf{B}^{(1)} = \begin{bmatrix}\mathbf{B}_1 \\ \mathbf{0} \\ \mathbf{0}\end{bmatrix}$$

Iteration 2: matching two moments at DC

- $\mathbf{C}_{21}^{(1)} = \underbrace{\mathbf{G}_{22}^{-1} \begin{bmatrix} \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \mathbf{M}_1^{(1)}}_{\text{from iteration 1}} \rightarrow$ QR: $\mathbf{C}_{21}^{(1)} = \begin{bmatrix} \mathbf{Q}_2^{(2)} & \mathbf{Q}_3^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{21}^{(2)} \\ \mathbf{0} \end{bmatrix} = \mathbf{Q}_2^{(2)} \mathbf{C}_{21}^{(2)}$
- $\mathbf{M}^{(2)} = \left[\begin{array}{c|c|c} \mathbf{M}_1^{(1)} & \begin{array}{c} \mathbf{0} \\ \mathbf{M}_2^{(2)} \end{array} & \begin{array}{c} \mathbf{0} \\ \mathbf{M}_3^{(2)} \end{array} \end{array} \right] = \left[\begin{array}{c|c|c} \mathbf{M}_1^{(1)} & \begin{array}{c} \mathbf{0} \\ \mathbf{Q}_2^{(2)} (\mathcal{G}^{(2)})^{-T} \end{array} & \begin{array}{c} \mathbf{0} \\ \mathbf{G}_{22}^{-T} \mathbf{Q}_3^{(2)} \end{array} \end{array} \right]$

$$\mathbf{G}^{(1)} = \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{33}^{(2)} \end{bmatrix} \quad \mathbf{C}^{(1)} = \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} & \mathbf{0} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} & \mathbf{C}_{32}^{(2)T} \\ \mathbf{0} & \mathbf{C}_{32}^{(2)} & \mathbf{C}_{33}^{(2)} \end{bmatrix} \quad \mathbf{B}^{(1)} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

Iteration 2: matching two moments at DC

- $\mathbf{C}_{21}^{(1)} = \underbrace{\mathbf{G}_{22}^{-1}\left[\mathbf{C}_{21} \quad \mathbf{C}_{22}\right]\mathbf{M}_1^{(1)}}_{\text{from iteration 1}} \rightarrow$ QR: $\mathbf{C}_{21}^{(1)} = \left[\mathbf{Q}_2^{(2)} \quad \mathbf{Q}_3^{(2)}\right]\begin{bmatrix}\mathbf{C}_{21}^{(2)} \\ \mathbf{0}\end{bmatrix} = \mathbf{Q}_2^{(2)}\mathbf{C}_{21}^{(2)}$
- $\mathbf{M}^{(2)} = \left[\begin{array}{c|c|c}\mathbf{M}_1^{(1)} & \begin{matrix}\mathbf{0}\\ \mathbf{M}_2^{(2)}\end{matrix} & \begin{matrix}\mathbf{0}\\ \mathbf{M}_3^{(2)}\end{matrix}\end{array}\right] = \left[\begin{array}{c|c|c}\mathbf{M}_1^{(1)} & \begin{matrix}\mathbf{0}\\ \mathbf{Q}_2^{(2)}(\boldsymbol{\mathcal{G}}^{(2)})^{-T}\end{matrix} & \begin{matrix}\mathbf{0}\\ \mathbf{G}_{22}^{-T}\mathbf{Q}_3^{(2)}\end{matrix}\end{array}\right]$

$$\mathbf{G}^{(1)} = \begin{bmatrix}\mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{33}^{(2)}\end{bmatrix} \quad \mathbf{C}^{(1)} = \begin{bmatrix}\mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} & \mathbf{0} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} & \mathbf{C}_{32}^{(2)T} \\ \mathbf{0} & \mathbf{C}_{32}^{(2)} & \mathbf{C}_{33}^{(2)}\end{bmatrix} \quad \mathbf{B}^{(1)} = \begin{bmatrix}\mathbf{B}_1 \\ \mathbf{0} \\ \mathbf{0}\end{bmatrix}$$

Iteration 2: matching two moments at DC

- $\mathbf{C}_{21}^{(1)} = \underbrace{\mathbf{G}_{22}^{-1} \begin{bmatrix} \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \mathbf{M}_1^{(1)}}_{\text{from iteration 1}} \rightarrow$ QR: $\mathbf{C}_{21}^{(1)} = \begin{bmatrix} \mathbf{Q}_2^{(2)} & \mathbf{Q}_3^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{21}^{(2)} \\ \mathbf{0} \end{bmatrix} = \mathbf{Q}_2^{(2)} \mathbf{C}_{21}^{(2)}$
- $\mathbf{M}^{(2)} = \left[\begin{array}{c|c|c} \mathbf{M}_1^{(1)} & \mathbf{0} & \mathbf{0} \\ & \mathbf{M}_2^{(2)} & \mathbf{M}_3^{(2)} \end{array} \right] = \left[\begin{array}{c|c|c} \mathbf{M}_1^{(1)} & \mathbf{0} & \mathbf{0} \\ & \mathbf{Q}_2^{(2)} (\mathcal{G}^{(2)})^{-T} & \mathbf{G}_{22}^{-T} \mathbf{Q}_3^{(2)} \end{array} \right]$

$$\mathbf{G}^{(1)} = \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{33}^{(2)} \end{bmatrix} \quad \mathbf{C}^{(1)} = \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} & \mathbf{0} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} & \mathbf{C}_{32}^{(2)T} \\ \mathbf{0} & \mathbf{C}_{32}^{(2)} & \mathbf{C}_{33}^{(2)} \end{bmatrix} \quad \mathbf{B}^{(1)} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

Iteration 2: matching two moments at DC

- $\mathbf{C}_{21}^{(1)} = \underbrace{\mathrm{G}_{22}^{-1}\begin{bmatrix}\mathbf{C}_{21} & \mathbf{C}_{22}\end{bmatrix}\mathbf{M}_1^{(1)}}_{\text{from iteration 1}} \rightarrow$ QR: $\mathbf{C}_{21}^{(1)} = \begin{bmatrix}\mathbf{Q}_2^{(2)} & \mathbf{Q}_3^{(2)}\end{bmatrix}\begin{bmatrix}\mathbf{C}_{21}^{(2)} \\ \mathbf{0}\end{bmatrix} = \mathbf{Q}_2^{(2)}\mathbf{C}_{21}^{(2)}$
- $\mathbf{M}^{(2)} = \left[\begin{array}{c|c|c}\mathrm{M}_1^{(1)} & \begin{matrix}0 \\ \mathbf{M}_2^{(2)}\end{matrix} & \begin{matrix}0 \\ \mathbf{M}_3^{(2)}\end{matrix}\end{array}\right] = \left[\begin{array}{c|c|c}\mathrm{M}_1^{(1)} & \begin{matrix}0 \\ \mathbf{Q}_2^{(2)}(\mathcal{G}^{(2)})^{-T}\end{matrix} & \begin{matrix}0 \\ \mathrm{G}_{22}^{-T}\mathbf{Q}_3^{(2)}\end{matrix}\end{array}\right]$

$$\mathbf{G}^{(1)} = \begin{bmatrix}\mathrm{G}_{11}^{(1)} & \mathrm{G}_{12}^{(2)} & \mathrm{G}_{12}^{(2)} \\ \mathbf{0} & \mathrm{G}_{22}^{(2)} & \mathrm{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathrm{G}_{33}^{(2)}\end{bmatrix} \quad \mathbf{C}^{(1)} = \begin{bmatrix}\mathrm{C}_{11}^{(1)} & \mathrm{C}_{21}^{(2)T} & \mathbf{0} \\ \mathbf{C}_{21}^{(2)} & \mathrm{C}_{22}^{(2)} & \mathrm{C}_{32}^{(2)T} \\ \mathbf{0} & \mathrm{C}_{32}^{(2)} & \mathrm{C}_{33}^{(2)}\end{bmatrix} \quad \mathbf{B}^{(1)} = \begin{bmatrix}\mathrm{B}_1 \\ \mathbf{0} \\ \mathbf{0}\end{bmatrix}$$

Iteration 2: matching two moments at DC

- $\mathbf{C}_{21}^{(1)} = \underbrace{\mathbf{G}_{22}^{-1}\begin{bmatrix}\mathbf{C}_{21} & \mathbf{C}_{22}\end{bmatrix}\mathbf{M}_1^{(1)}}_{\text{from iteration 1}} \rightarrow$ QR: $\mathbf{C}_{21}^{(1)} = \begin{bmatrix}\mathbf{Q}_2^{(2)} & \mathbf{Q}_3^{(2)}\end{bmatrix}\begin{bmatrix}\mathbf{C}_{21}^{(2)} \\ \mathbf{0}\end{bmatrix} = \mathbf{Q}_2^{(2)}\mathbf{C}_{21}^{(2)}$
- $\mathbf{M}^{(2)} = \left[\begin{array}{c|c|c}\mathbf{M}_1^{(1)} & \begin{matrix}\mathbf{0}\\ \mathbf{M}_2^{(2)}\end{matrix} & \begin{matrix}\mathbf{0}\\ \mathbf{M}_3^{(2)}\end{matrix}\end{array}\right] = \left[\begin{array}{c|c|c}\mathbf{M}_1^{(1)} & \begin{matrix}\mathbf{0}\\ \mathbf{Q}_2^{(2)}(\boldsymbol{\mathcal{G}}^{(2)})^{-T}\end{matrix} & \begin{matrix}\mathbf{0}\\ \mathbf{G}_{22}^{-T}\mathbf{Q}_3^{(2)}\end{matrix}\end{array}\right]$

where $\boldsymbol{\mathcal{G}}^{(2)} = \mathbf{Q}_2^{(2)T}\mathbf{G}_{22}\mathbf{Q}_2^{(2)}$

$$\mathbf{G}^{(1)} = \begin{bmatrix}\mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{33}^{(2)}\end{bmatrix} \quad \mathbf{C}^{(1)} = \begin{bmatrix}\mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} & \mathbf{0} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} & \mathbf{C}_{32}^{(2)T} \\ \mathbf{0} & \mathbf{C}_{32}^{(2)} & \mathbf{C}_{33}^{(2)}\end{bmatrix} \quad \mathbf{B}^{(1)} = \begin{bmatrix}\mathbf{B}_1 \\ \mathbf{0} \\ \mathbf{0}\end{bmatrix}$$

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Iteration 2: matching two moments at DC

- $\mathbf{C}_{21}^{(1)} = \underbrace{\mathbf{G}_{22}^{-1}\left[\mathbf{C}_{21} \quad \mathbf{C}_{22}\right]\mathbf{M}_1^{(1)}}_{\text{from iteration 1}} \rightarrow$ QR: $\mathbf{C}_{21}^{(1)} = \left[\mathbf{Q}_2^{(2)} \quad \mathbf{Q}_3^{(2)}\right]\begin{bmatrix}\mathbf{C}_{21}^{(2)} \\ \mathbf{0}\end{bmatrix} = \mathbf{Q}_2^{(2)}\mathbf{C}_{21}^{(2)}$
- $\mathbf{M}^{(2)} = \left[\mathbf{M}_1^{(1)} \;\middle|\; \begin{matrix}\mathbf{0} \\ \mathbf{M}_2^{(2)}\end{matrix} \;\middle|\; \begin{matrix}\mathbf{0} \\ \mathbf{M}_3^{(2)}\end{matrix}\right] = \left[\mathbf{M}_1^{(1)} \;\middle|\; \begin{matrix}\mathbf{0} \\ \mathbf{Q}_2^{(2)}(\boldsymbol{\mathcal{G}}^{(2)})^{-T}\end{matrix} \;\middle|\; \begin{matrix}\mathbf{0} \\ \mathbf{G}_{22}^{-T}\mathbf{Q}_3^{(2)}\end{matrix}\right]$

  where $\boldsymbol{\mathcal{G}}^{(2)} = \mathbf{Q}_2^{(2)T}\mathbf{G}_{22}\mathbf{Q}_2^{(2)}$

$$\mathbf{G}^{(1)} = \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{33}^{(2)} \end{bmatrix} \quad \mathbf{C}^{(1)} = \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} & \mathbf{0} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} & \mathbf{C}_{32}^{(2)T} \\ \mathbf{0} & \mathbf{C}_{32}^{(2)} & \mathbf{C}_{33}^{(2)} \end{bmatrix} \quad \mathbf{B}^{(1)} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

Iteration 2: matching two moments at DC

- $\mathbf{C}_{21}^{(1)} = \underbrace{\mathbf{G}_{22}^{-1}\left[\mathbf{C}_{21} \quad \mathbf{C}_{22}\right]\mathbf{M}_1^{(1)}}_{\text{from iteration 1}} \rightarrow$ QR: $\mathbf{C}_{21}^{(1)} = \left[\mathbf{Q}_2^{(2)} \quad \mathbf{Q}_3^{(2)}\right]\begin{bmatrix}\mathbf{C}_{21}^{(2)} \\ \mathbf{0}\end{bmatrix} = \mathbf{Q}_2^{(2)}\mathbf{C}_{21}^{(2)}$
- $\mathbf{M}^{(2)} = \left[\begin{array}{c|c|c} \mathbf{M}_1^{(1)} & \begin{matrix}\mathbf{0} \\ \mathbf{M}_2^{(2)}\end{matrix} & \begin{matrix}\mathbf{0} \\ \mathbf{M}_3^{(2)}\end{matrix}\end{array}\right] = \left[\begin{array}{c|c|c} \mathbf{M}_1^{(1)} & \begin{matrix}\mathbf{0} \\ \mathbf{Q}_2^{(2)}(\boldsymbol{\mathcal{G}}^{(2)})^{-T}\end{matrix} & \begin{matrix}\mathbf{0} \\ \mathbf{G}_{22}^{-T}\mathbf{Q}_3^{(2)}\end{matrix}\end{array}\right]$

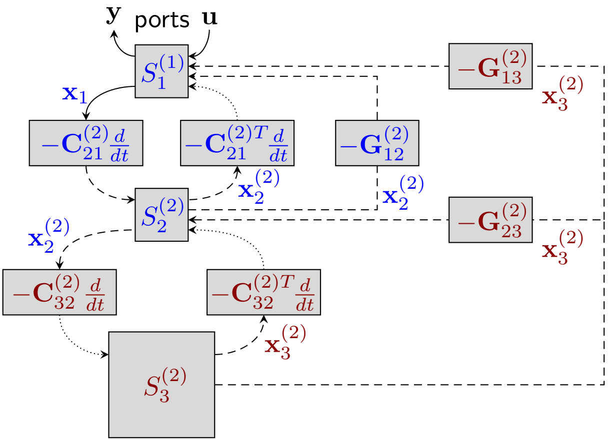
  where $\boldsymbol{\mathcal{G}}^{(2)} = \mathbf{Q}_2^{(2)T}\mathbf{G}_{22}\mathbf{Q}_2^{(2)}$

$$\mathbf{G}^{(1)} = \begin{bmatrix}\mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{33}^{(2)}\end{bmatrix} \quad \mathbf{C}^{(1)} = \begin{bmatrix}\mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} & \mathbf{0} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} & \mathbf{C}_{32}^{(2)T} \\ \mathbf{0} & \mathbf{C}_{32}^{(2)} & \mathbf{C}_{33}^{(2)}\end{bmatrix} \quad \mathbf{B}^{(1)} = \begin{bmatrix}\mathbf{B}_1 \\ \mathbf{0} \\ \mathbf{0}\end{bmatrix}$$

Iteration 2: matching two moments at DC

$$\begin{matrix} S_1^{(1)}: \\ S_2^{(2)}: \\ S_3^{(2)}: \end{matrix} \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{33}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(2)} \\ \mathbf{x}_3^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} & \mathbf{0} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} & \mathbf{C}_{32}^{(2)T} \\ \mathbf{0} & \mathbf{C}_{32}^{(2)} & \mathbf{C}_{33}^{(2)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(2)} \\ \dot{\mathbf{x}}_3^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

Connection to the next subsystem is only through $\mathbf{C}_{21}^{(2)} \frac{d}{dt}$ or $\mathbf{C}_{32}^{(2)} \frac{d}{dt}$



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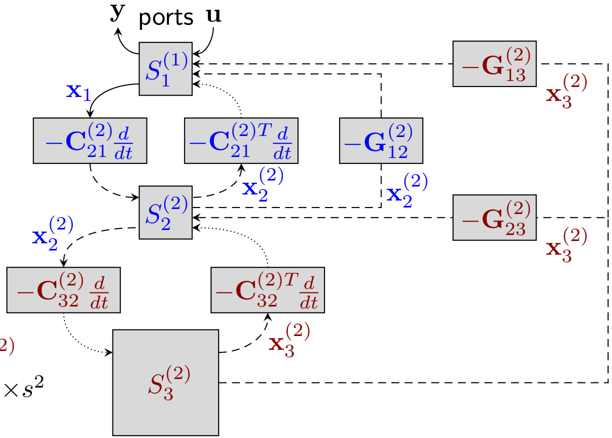
Iteration 2: matching two moments at DC

$$
\begin{matrix} S_1^{(1)}: \\ S_2^{(2)}: \\ S_3^{(2)}: \end{matrix}
\begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{33}^{(2)} \end{bmatrix}
\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(2)} \\ \mathbf{x}_3^{(2)} \end{bmatrix}
+
\begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} & \mathbf{0} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} & \mathbf{C}_{32}^{(2)T} \\ \mathbf{0} & \mathbf{C}_{32}^{(2)} & \mathbf{C}_{33}^{(2)} \end{bmatrix}
\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(2)} \\ \dot{\mathbf{x}}_3^{(2)} \end{bmatrix}
=
\begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u}
$$

Connection to the next subsystem is only through $\mathbf{C}_{21}^{(2)} \frac{d}{dt}$ or $\mathbf{C}_{32}^{(2)} \frac{d}{dt}$



Input to $S_3^{(2)}$ went through $\times s^2$

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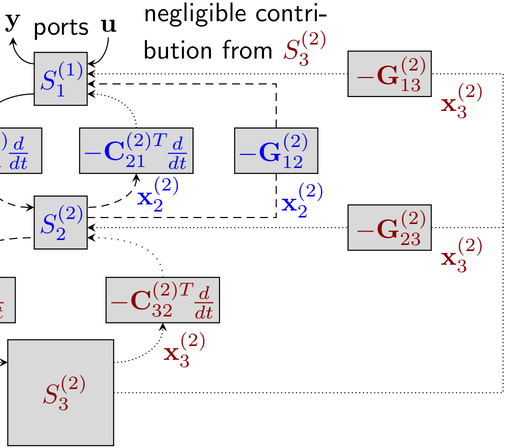
Iteration 2: matching two moments at DC

$$\begin{matrix} S_1^{(1)}: \\ S_2^{(2)}: \\ S_3^{(2)}: \end{matrix} \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{33}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(2)} \\ \mathbf{x}_3^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} & \mathbf{0} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} & \mathbf{C}_{32}^{(2)T} \\ \mathbf{0} & \mathbf{C}_{32}^{(2)} & \mathbf{C}_{33}^{(2)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(2)} \\ \dot{\mathbf{x}}_3^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

Connection to the next subsystem is only through $\mathbf{C}_{21}^{(2)} \frac{d}{dt}$ or $\mathbf{C}_{32}^{(2)} \frac{d}{dt}$

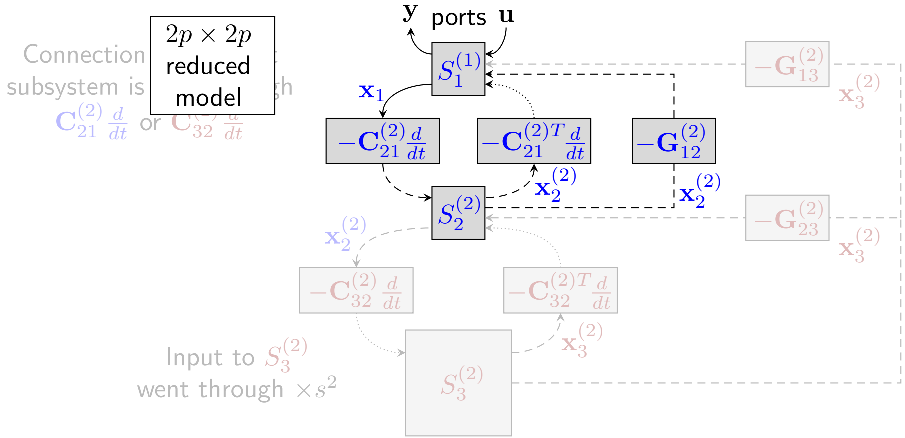


Input to $S_3^{(2)}$ went through $\times s^2$

68

Iteration 2: matching two moments at DC

$$\begin{array}{l} S_1^{(1)}: \\ S_2^{(2)}: \\ S_3^{(2)}: \end{array} \begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{33}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(2)} \\ \mathbf{x}_3^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} & \mathbf{0} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} & \mathbf{C}_{32}^{(2)T} \\ \mathbf{0} & \mathbf{C}_{32}^{(2)} & \mathbf{C}_{33}^{(2)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(2)} \\ \dot{\mathbf{x}}_3^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$



Iteration 2: computation steps

Reduced system for $q = 2$ (completed computations)

$$\begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

Iteration 2: computation steps

Reduced system for $q = 2$ (completed computations)

$$\begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

1. QR factorization: $\mathbf{C}_{21}^{(1)} = \mathbf{G}_{22}^{-1} \begin{bmatrix} \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \mathbf{M}_1^{(1)} = \mathbf{Q}_2^{(2)} \mathbf{C}_{21}^{(2)}$

Iteration 2: computation steps

Reduced system for $q = 2$ (completed computations)

$$\begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

1. QR factorization: $\mathbf{C}_{21}^{(1)} = \mathbf{G}_{22}^{-1} \begin{bmatrix} \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \mathbf{M}_1^{(1)} = \mathbf{Q}_2^{(2)} \mathbf{C}_{21}^{(2)}$

Iteration 2: computation steps

Reduced system for $q = 2$ (completed computations)

$$\begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

1. QR factorization: $\mathbf{C}_{21}^{(1)} = \mathbf{G}_{22}^{-1} \begin{bmatrix} \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \mathbf{M}_1^{(1)} = \mathbf{Q}_2^{(2)} \mathbf{C}_{21}^{(2)}$
2. Compute $\boldsymbol{\mathcal{G}}^{(2)} = \mathbf{Q}_2^{(2)T} \mathbf{G}_{22} \mathbf{Q}_2^{(2)}$

Iteration 2: computation steps

Reduced system for $q = 2$ (completed computations)

$$\begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

1. QR factorization: $\mathbf{C}_{21}^{(1)} = \mathbf{G}_{22}^{-1} \begin{bmatrix} \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \mathbf{M}_1^{(1)} = \mathbf{Q}_2^{(2)} \mathbf{C}_{21}^{(2)}$
2. Compute $\boldsymbol{\mathcal{G}}^{(2)} = \mathbf{Q}_2^{(2)T} \mathbf{G}_{22} \mathbf{Q}_2^{(2)}$
3. Compute $\mathbf{M}_2^{(2)} = \mathbf{Q}_2^{(2)} (\boldsymbol{\mathcal{G}}^{(2)})^{-T}$. Recall: $\mathbf{M}^{(2)} = \left[\begin{array}{c|c|c} \mathbf{M}_1^{(1)} & \begin{matrix} \mathbf{0} \\ \mathbf{M}_2^{(2)} \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{M}_3^{(2)} \end{matrix} \end{array} \right]$

Iteration 2: computation steps

Reduced system for $q = 2$ (completed computations)

$$\begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

1. QR factorization: $\mathbf{C}_{21}^{(1)} = \mathbf{G}_{22}^{-1} \begin{bmatrix} \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \mathbf{M}_1^{(1)} = \mathbf{Q}_2^{(2)} \mathbf{C}_{21}^{(2)}$
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4. Compute $\mathbf{G}_{12}^{(2)} = \mathbf{M}_1^{(1)T} \mathbf{G} \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_2^{(2)} \end{bmatrix}$

Iteration 2: computation steps

Reduced system for $q = 2$ (completed computations)

$$\begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

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5. Compute $\mathbf{C}_{22}^{(2)} = \begin{bmatrix} \mathbf{0} & \mathbf{M}_2^{(2)T} \end{bmatrix} \mathbf{C} \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_2^{(2)} \end{bmatrix}$

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Reduced system for $q = 2$ (completed computations)

$$\begin{bmatrix} \mathbf{G}_{11}^{(1)} & \mathbf{G}_{12}^{(2)} \\ \mathbf{0} & \mathbf{G}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11}^{(1)} & \mathbf{C}_{21}^{(2)T} \\ \mathbf{C}_{21}^{(2)} & \mathbf{C}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

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6. Result of simplification: $\mathbf{G}_{22}^{(2)} = (\boldsymbol{\mathcal{G}}^{(2)})^{-T}$

- PRIMA: 2 moments $\rightarrow 2p$ orthogonal vectors in $\mathbf{V}$ (p = port count)
- Proposed: 2 moments $\rightarrow$ p orthogonal vectors in $\mathbf{Q}_2^{(2)}$

Transformed system (general case)

$$\left[\begin{array}{cccc|c} \mathbf{G}_{11}^{(1)} & \cdots & \mathbf{G}_{1q}^{(q)} & & \mathbf{G}_{1,q+1}^{(q)} \\ \mathbf{0} & \ddots & \vdots & & \vdots \\ \vdots & \ddots & \mathbf{G}_{qq}^{(q)} & & \mathbf{G}_{q,q+1}^{(k)} \\ \hline \mathbf{0} & \cdots & \mathbf{0} & & \mathbf{G}_{q+1,q+1}^{(q)} \end{array}\right] \left[\begin{array}{c} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_q^{(q)} \\ \hline \mathbf{x}_{q+1}^{(q)} \end{array}\right] +$$

$$+ \left[\begin{array}{ccc|c} \mathbf{C}_{11}^{(1)} & * & & \\ \mathbf{C}_{21}^{(2)} & \ddots & \ddots & \\ & \ddots & \mathbf{C}_{qq}^{(q)} & * \\ \hline & & \mathbf{C}_{q+1,q}^{(q)} & \mathbf{C}_{q+1,q+1}^{(q)} \end{array}\right] \left[\begin{array}{c} \dot{\mathbf{x}}_1 \\ \vdots \\ \dot{\mathbf{x}}_q^{(q)} \\ \hline \dot{\mathbf{x}}_{q+1}^{(q)} \end{array}\right] = \left[\begin{array}{c} \mathbf{B}_1 \\ \mathbf{0} \\ \vdots \\ \hline \mathbf{0} \end{array}\right] \mathbf{u}$$

Transformed system (general case)

$$
\left[\begin{array}{cccc:c}
\mathbf{G}_{11}^{(1)} & \cdots & \mathbf{G}_{1q}^{(q)} & & \mathbf{G}_{1,q+1}^{(q)} \\
\mathbf{0} & \ddots & \vdots & & \vdots \\
\vdots & \ddots & \mathbf{G}_{qq}^{(q)} & & \mathbf{G}_{q,q+1}^{(k)} \\
\hdashline
\mathbf{0} & \cdots & \mathbf{0} & & \mathbf{G}_{q+1,q+1}^{(q)}
\end{array}\right]
\left[\begin{array}{c}
\mathbf{x}_1 \\ \vdots \\ \mathbf{x}_q^{(q)} \\ \hdashline \mathbf{x}_{q+1}^{(q)}
\end{array}\right] +
$$

$$
+ \left[\begin{array}{ccc:c}
\mathbf{C}_{11}^{(1)} & * & & \\
\mathbf{C}_{21}^{(2)} & \ddots & \ddots & \\
 & \ddots & \mathbf{C}_{qq}^{(q)} & * \\
\hdashline
 & & \mathbf{C}_{q+1,q}^{(q)} & \mathbf{C}_{q+1,q+1}^{(q)}
\end{array}\right]
\left[\begin{array}{c}
\dot{\mathbf{x}}_1 \\ \vdots \\ \dot{\mathbf{x}}_q^{(q)} \\ \hdashline \dot{\mathbf{x}}_{q+1}^{(q)}
\end{array}\right] =
\left[\begin{array}{c}
\mathbf{B}_1 \\ \mathbf{0} \\ \vdots \\ \hdashline \mathbf{0}
\end{array}\right] \mathbf{u}
$$

- Connection to the next system is only through $\mathbf{C}_{21}^{(2)}, \ldots, \mathbf{C}_{q+1,q}^{(q)}$ block

80

Transformed system (general case)

$$
\left[\begin{array}{ccc|c}
\mathbf{G}_{11}^{(1)} & \cdots & \mathbf{G}_{1q}^{(q)} & \mathbf{G}_{1,q+1}^{(q)} \\
\mathbf{0} & \ddots & \vdots & \vdots \\
\vdots & \ddots & \mathbf{G}_{qq}^{(q)} & \mathbf{G}_{q,q+1}^{(k)} \\
\hline
\mathbf{0} & \cdots & \mathbf{0} & \mathbf{G}_{q+1,q+1}^{(q)}
\end{array}\right]
\left[\begin{array}{c}
\mathbf{x}_1 \\
\vdots \\
\mathbf{x}_q^{(q)} \\
\hline
\mathbf{x}_{q+1}^{(q)}
\end{array}\right]
+
$$

$$
+
\left[\begin{array}{ccc|c}
\mathbf{C}_{11}^{(1)} & * & & \\
\mathbf{C}_{21}^{(2)} & \ddots & \ddots & \\
 & \ddots & \mathbf{C}_{qq}^{(q)} & * \\
\hline
 & & \mathbf{C}_{q+1,q}^{(q)} & \mathbf{C}_{q+1,q+1}^{(q)}
\end{array}\right]
\left[\begin{array}{c}
\dot{\mathbf{x}}_1 \\
\vdots \\
\dot{\mathbf{x}}_q^{(q)} \\
\hline
\dot{\mathbf{x}}_{q+1}^{(q)}
\end{array}\right]
=
\left[\begin{array}{c}
\mathbf{B}_1 \\
\mathbf{0} \\
\vdots \\
\hline
\mathbf{0}
\end{array}\right]
\mathbf{u}
$$

- Connection to the next system is only through $\mathbf{C}_{21}^{(2)}, \ldots, \mathbf{C}_{q+1,q}^{(q)}$ block
- This creates a cascade of systems with progressively weaker excitation

Transformed system (general case)

$$\left[\begin{array}{ccc:c} \mathbf{G}_{11}^{(1)} & \dots & \mathbf{G}_{1q}^{(q)} & \mathbf{G}_{1,q+1}^{(q)} \\ \mathbf{0} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \mathbf{G}_{qq}^{(q)} & \mathbf{G}_{q,q+1}^{(k)} \\ \hdashline \mathbf{0} & \dots & \mathbf{0} & \mathbf{G}_{q+1,q+1}^{(q)} \end{array}\right] \left[\begin{array}{c} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_q^{(q)} \\ \hdashline \mathbf{x}_{q+1}^{(q)} \end{array}\right] + \left[\begin{array}{ccc:c} \mathbf{C}_{11}^{(1)} & * & & \\ \mathbf{C}_{21}^{(2)} & \ddots & \ddots & \\ & \ddots & \mathbf{C}_{qq}^{(q)} & * \\ \hdashline & & \mathbf{C}_{q+1,q}^{(q)} & \mathbf{C}_{q+1,q+1}^{(q)} \end{array}\right] \left[\begin{array}{c} \dot{\mathbf{x}}_1 \\ \vdots \\ \dot{\mathbf{x}}_q^{(q)} \\ \hdashline \dot{\mathbf{x}}_{q+1}^{(q)} \end{array}\right] = \left[\begin{array}{c} \mathbf{B}_1 \\ \mathbf{0} \\ \vdots \\ \hdashline \mathbf{0} \end{array}\right] \mathbf{u}$$

- Connection to the next system is only through $\mathbf{C}_{21}^{(2)}, \dots, \mathbf{C}_{q+1,q}^{(q)}$ block
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- The last system (large in size) can be neglected $\rightarrow$ $qp \times qp$ reduced model

Transformed system (general case)

$$\left[\begin{array}{cccc:c} \mathbf{G}_{11}^{(1)} & \dots & \mathbf{G}_{1q}^{(q)} & & \mathbf{G}_{1,q+1}^{(q)} \\ \mathbf{0} & \ddots & \vdots & & \vdots \\ \vdots & \ddots & \mathbf{G}_{qq}^{(q)} & & \mathbf{G}_{q,q+1}^{(k)} \\ \hdashline \mathbf{0} & \dots & \mathbf{0} & & \mathbf{G}_{q+1,q+1}^{(q)} \end{array}\right] \left[\begin{array}{c} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_q^{(q)} \\ \hdashline \mathbf{x}_{q+1}^{(q)} \end{array}\right] +$$

$$+ \left[\begin{array}{ccc:c} \mathbf{C}_{11}^{(1)} & * & & \\ \mathbf{C}_{21}^{(2)} & \ddots & \ddots & \\ & \ddots & \mathbf{C}_{qq}^{(q)} & * \\ \hdashline & & \mathbf{C}_{q+1,q}^{(q)} & \mathbf{C}_{q+1,q+1}^{(q)} \end{array}\right] \left[\begin{array}{c} \dot{\mathbf{x}}_1 \\ \vdots \\ \dot{\mathbf{x}}_q^{(q)} \\ \hdashline \dot{\mathbf{x}}_{q+1}^{(q)} \end{array}\right] = \left[\begin{array}{c} \mathbf{B}_1 \\ \mathbf{0} \\ \vdots \\ \hdashline \mathbf{0} \end{array}\right] \mathbf{u}$$

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- This creates a cascade of systems with progressively weaker excitation
- The last system (large in size) can be neglected $\rightarrow qp \times qp$ reduced model
- Provably passive

Transformed system (general case)

$$
\left[\begin{array}{cccc|c}
\mathbf{G}_{11}^{(1)} & \dots & \mathbf{G}_{1q}^{(q)} & & \mathbf{G}_{1,q+1}^{(q)} \\
\mathbf{0} & \ddots & \vdots & & \vdots \\
\vdots & \ddots & \mathbf{G}_{qq}^{(q)} & & \mathbf{G}_{q,q+1}^{(k)} \\
\hline
\mathbf{0} & \dots & \mathbf{0} & & \mathbf{G}_{q+1,q+1}^{(q)}
\end{array}\right]
\left[\begin{array}{c}
\mathbf{x}_1 \\ \vdots \\ \mathbf{x}_q^{(q)} \\ \hline \mathbf{x}_{q+1}^{(q)}
\end{array}\right]
+
\left[\begin{array}{ccc|c}
\mathbf{C}_{11}^{(1)} & * & & \\
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 & \ddots & \mathbf{C}_{qq}^{(q)} & * \\
\hline
 & & \mathbf{C}_{q+1,q}^{(q)} & \mathbf{C}_{q+1,q+1}^{(q)}
\end{array}\right]
\left[\begin{array}{c}
\dot{\mathbf{x}}_1 \\ \vdots \\ \dot{\mathbf{x}}_q^{(q)} \\ \hline \dot{\mathbf{x}}_{q+1}^{(q)}
\end{array}\right]
=
\left[\begin{array}{c}
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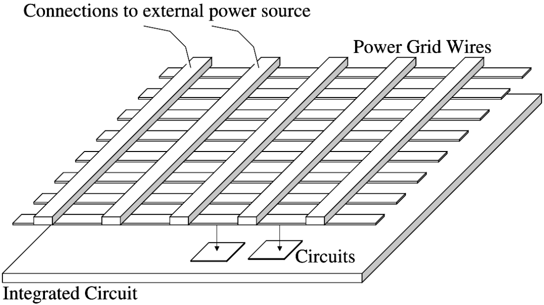
\end{array}\right]
\mathbf{u}
$$

- Connection to the next system is only through $\mathbf{C}_{21}^{(2)}, \dots, \mathbf{C}_{q+1,q}^{(q)}$ block
- This creates a cascade of systems with progressively weaker excitation
- The last system (large in size) can be neglected $\rightarrow$ $qp \times qp$ reduced model
- Provably passive
- Matches q moments at DC

IBM power grid benchmarks

ibmpg1t and ibmpg2t [Z. Li, P. Li, & S. R. Nassif; 2011]



Small part of a typical benchmark [Nassif; 2008]

Benchmark	Nodes	R	C	L	p
ibmpg1t	25k	41k	11k	277	V: 100, I: 9k
ibmpg2t	164k	245k	37k	330	V: 120, I: 37k

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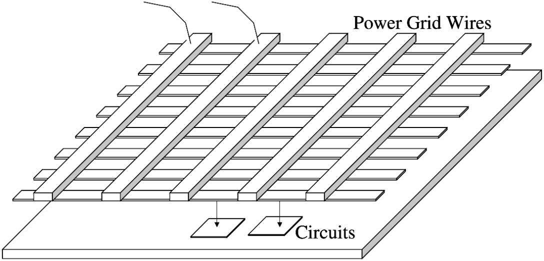

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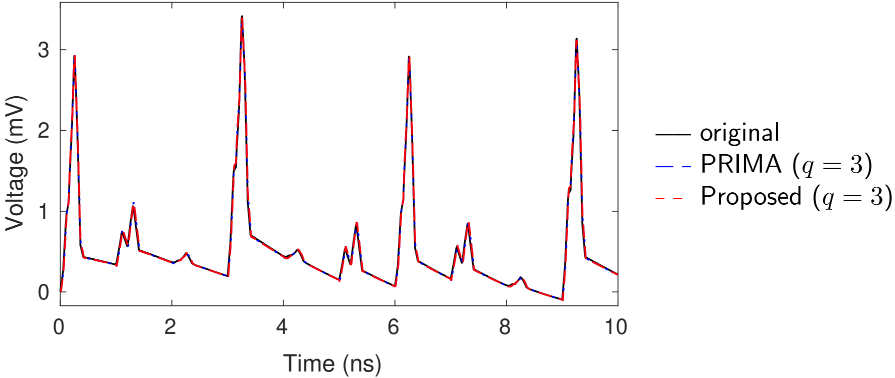
We select different subsets of these ports.

Test 1: accuracy of reduced model

Benchmark: ibmpg2t ($n = 164$k, $p = 750$)

Waveforms at node n0_3968_6546 (output with worst case error)



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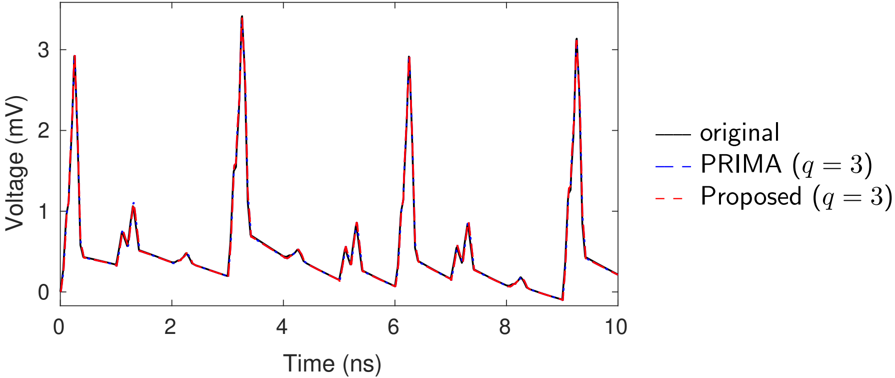

Error PRIMA vs original: 4.88%
Error Proposed vs original: 4.88%

Test 2: reduction time vs number of iterations

Benchmark: ibmpg2t ($n = 164$k, $p = 900$)

q	Reduction time		Speedup
	PRIMA	Proposed	
1	31.1 s	12.2 s	$\times$2.55
2	84.0 s	56.7 s	$\times$1.48
3	143.4 s	119.2 s	$\times$1.20
4	225.0 s	205.2 s	$\times$1.10

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We achieve some speedup for small q.
But the speedup tends to decrease when q increases.

Test 3: reduction for acceptable error

- Benchmarks: ibmpg1t and ibmpg2t with different p
- q selected to bring the error at each port below 5%

p	q	Reduction time		Speedup	Error	
		PRIMA	Proposed		PRIMA	Proposed
ibmpg1t (original states: 26k)						
183	7	5.6 s	7.8 s	$\times$0.72	2.2%	2.2%
477	5	14.2 s	14.3 s	$\times$0.99	0.2%	0.2%
717	4	19.4 s	17.3 s	$\times$1.12	0.3%	0.3%
847	3	15.5 s	12.2 s	$\times$1.27	2.6%	2.6%
ibmpg2t (original states: 164k)						
200	6	53.6 s	61.1 s	$\times$0.88	3.4%	3.4%
500	5	137.4 s	137.2 s	$\times$1.00	0.2%	0.2%
750	3	117.0 s	90.8 s	$\times$1.29	4.9%	4.9%
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- Speedups are very modest if they happen – *still work in progress*
- We hope that the proposed ideas could be useful for achieving efficient reduction of RLC parasitics.

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- We proposed an extension of TurboMOR-RC to RLC circuits
  - Non-trivial: the assumptions on the properties of $\mathbf{G}$ are violated
- The algorithm can provide some speedup if q is not too high
- Tends to happen for large p
- Interesting property: large p is a difficult case for state of the art methods
- Speedups are very modest if they happen – *still work in progress*
- We hope that the proposed ideas could be useful for achieving efficient reduction of RLC parasitics.

Thank you!

Acknowledgements:

- Canada Research Chairs program
- Ontario Early Researcher Award program
- NSERC Postgraduate Scholarships-Doctoral Program